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Duality Cascade and Oblique Phases in Non-Commutative Open String Theory

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Abstract

We investigate the complete phase diagram of the decoupled world-sheet theory of (P, Q) strings. These theories include 1+1 dimensional super Yang-Mills theory and non-commutative open string theory. We find that the system exhibits a rich fractal phase structure, including a cascade of alternating supergravity, gauge theory, and matrix string theory phases. The cascade proceeds via a series of $SL(2, \mathbf{Z})$ S-duality transformations, and depends sensitively on P and Q. In particular, we find that the system may undergo multiple Hagedorn-type transitions as the temperature is varied.

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1 Introduction

Duality is a powerful tool for analyzing dynamical aspects of field theories and string theories. Some dualities are exact. They assert that two seemingly different theories are physically equivalent. String theory with sufficiently many unbroken supersymmetries are believed to exhibit such exact duality relations, in the form of discrete symmetry relations acting on the space of couplings. These symmetries, and the fact that a large class of field theories can be formulated as decoupling limits of string theories, have been used to derive many examples of field theory dualities, as well as dual correspondences between field theories and supergravity theories. Similar considerations led to the understanding of duality relations among less familiar theories such as non-commutative gauge theories, non-commutative open string theories, and little string theories.

In this article, we consider the duality relations of non-commutative open string theories [**?**, **?**, **?**] in 1+1 dimensions with 16 supercharges. These theories can be formulated as a decoupling limit of bound states of D-strings and fundamental strings in type IIB string theory. The relevant duality relations follow from the $SL(2,\mathbf{Z})$ S-duality symmetry of type IIB string theory and the gauge theory/supergravity correspondence applied to (P,Q) strings. Our aim is to use a combination of both these dualities to gain insight into the thermodynamic properties of the theory. Various authors have also considered aspects of S-duality in the context of NCOS theories [**?**, **?**, **?**] and their supergravity duals[**?**, **?**].

NCOS theory is closely related to ordinary 1+1-d super Yang-Mills theory in at least two different ways. First, for rational value of its string coupling G_o^2, it's known to be S-dual to ordinary SYM theory with a non-zero electric flux, which therefore provides the proper ultraviolet definition of the theory. On the other hand, like any other known open string theory, NCOS theory reduces, at scales sufficiently below its string scale, to an effective low energy gauge theory. Near the NCOS string scale, however, this gauge theory description breaks down, and the system undergoes a phase transition into an effective matrix string theory phase [**?**, **?**]. The formation of the matrix strings can be viewed as an ionization process of the non-commutative open strings, that escape via the Coulomb branch from the bound state with the D-strings.

On general grounds, different dual descriptions are never simultaneously weakly coupled, since two distinct weakly coupled theories are manifestly inequivalent. This means that for given temperature and couplings, one can expect that, among the set of theories related by duality, there typically exists one preferred description which is most weakly coupled. In this paper, we will use this intuition to map out the complete phase diagram of the 1+1-dimensional NCOS theory. The main new conclusions of our study are the following:

- In most studies done so far of the thermodynamics of 1+1-dimensional NCOS theory, the effective open string coupling constant was assumed to be fixed at some rational value $G_o^2 = P/Q$. We will find, however, that G_o^2 can be any real positive number. This allows us to view NCOS theory as a continuous non-commutative deformation of ordinary gauge theory.

- Upon systematic consideration of the role of full $SL(2,\mathbf{Z})$ duality structure and the AdS/CFT correspondence, a remarkably elaborate phase structure emerges. Various $SL(2,\mathbf{Z})$ dual descriptions become preferred in disjoint regions of the phase diagram parameterized by temperature T and the NCOS coupling constant G_o^2. These regions form a complicated fractal pattern.

- As a function of T, the theory can go through a cascade of alternating supergravity, gauge theory, and matrix string theory phases. The cascade proceeds via a series of $SL(2,\mathbf{Z})$ S-duality transformations, and depends sensitively on P and Q. In particular, we find that the system may undergo a sequence of successive ionization and recombination transitions.

The fractal pattern seen in the phase diagram closely resembles the phase structure found for the supergravity duals of non-commutative Yang-Mills theories on a torus [**?**, **?**]. There, the role of $SL(2,\mathbf{Z})$ was played by the Morita/T-duality group. The duality cascade, which involved only the supergravity descriptions, were found not to give rise to any observable thermodynamic effects, simply because the area of the horizon in Einstein frame is invariant under T-duality transformations. Here, the duality cascade will act among the gauge theory, matrix theory, and supergravity phases, giving rise to thermodynamically observable cross-over effects.

The organization of this paper is as follows. In section 2, we collect various preliminary facts regarding NCOS theory as decoupling limit of string theory and the form of the $SL(2,\mathbf{Z})$ S-duality transformations. In section 3, we analyze the role of supergravity dual and the Hagedorn transition for the theory corresponding to a given set of charges P and Q. In section 4, we describe how the various dual descriptions fit together to form a continuous, though fractal, phase diagram. We conclude in section 5.

2 Preliminaries

2.1 Parameters of 1+1-d NCOS theory

In this subsection we introduce the parameter space of 1+1-d non-commutative open string theory.

Since 1+1-d NCOS theory is an interacting theory of open strings, it is specified by an open string coupling constant, G_o, and by the string tension, α'_{eff}. In addition, we can introduce $U(Q)$ Chan-Paton factors, as well as turn on a discrete electric flux. In two dimensions, this flux behaves like a discrete θ parameter [**?**]

$$\theta = \frac{2\pi P}{Q} \tag{2.1}$$

where P is an integer ranging from zero to $Q-1$. In the language of the underlying IIB string theory, P and Q count the number of fundamental and Dirichlet strings, respectively, that make up the bound state [**?**].

Just as in ordinary open string theory, we can expect that, in a suitable low energy regime, NCOS theory reduces to 1+1-d super Yang-Mills theory with $U(Q)$ gauge symmetry. The dimensionful gauge coupling g_{YM} is related to the NCOS coupling and string length via[1]

$$g_{YM}^2 = \frac{G_o^2}{\alpha'_{eff}} \ . \tag{2.2}$$

This infrared gauge theory should not be confused with the S-dual Yang-Mills theory in the ultraviolet of NCOS frequently discussed in the literature [**?**, **?**, **?**].

Since g_{YM}^2, α'_{eff}, and G_o^2 are related by (2.2), any two out of the three can be taken as the parameters defining the theory. For our purposes it will be convenient to think of NCOS theory as a modification of super Yang-Mills theory in the ultraviolet, induced by an irrelevant space-time non-commutativity perturbation. To emphasize this point of view, we will choose our parameters to be g_{YM}^2 and G_o^2. We will see later that, contrary to some claims in the literature [**?**, **?**], G_o^2 can in fact take on arbitrary positive real values, which from the gauge theory perspective sets the scale of the non-commutativity parameter, in units set by g_{YM}^2. Ordinary super Yang-Mills theory is recovered in the limit $G_o^2 \to 0$, with g_{YM} held fixed.

To summarize, the set of independent parameters that we will use to parameterize non-commutative open strings in 1+1 dimensions are the following:

$$\left\{ g_{YM}^2, \quad G_o^2, \quad P, \quad Q \right\} \ . \tag{2.3}$$

[1] Here and hereafter, we will ignore constant numerical factors of order one.

For the purpose of studying the action of $SL(2, \mathbf{Z})$ duality group on these parameters, it will sometimes be useful to separate the factor N which is the greatest common divisor of P and Q and write

$$P = Np, \qquad Q = Nq \tag{2.4}$$

where p and q are relatively prime integers.

2.2 SYM Decoupling limit of (P,Q) strings

Here we recall the decoupling limit of the (P, Q) string theory that produces 1+1-dimensional SYM theory, and introduce its supergravity dual.

Starting from the world sheet theory of a (P, Q) string bound state in IIB string theory, we can consider the limit

$$g_s \to 0 \tag{2.5}$$

while focusing on physics taking place at energy scale or temperature of order

$$T^2 = g_{_{YM}}^2 = \frac{g_s}{\alpha'} \ . \tag{2.6}$$

In this limit, the world sheet theory reduces to 1+1 dimensional super Yang-Mills theory with gauge coupling $g_{_{YM}}$, gauge group $U(Q)$ and P units of electric flux.

To formulate the corresponding near horizon supergravity geometry, we can start from the full IIB supergravity solution of the (P, Q) string obtained by Schwarz in [?]. This solution is parameterized by the asymptotic values of the string coupling and axion field, and by the two quantized charges P and Q. Applying the scaling limit (2.5), while focusing on the range of radial coordinates parameterized by $U = r/\alpha'$ with r as in [?], gives

$$\begin{aligned}
ds^2 &= \alpha' \left(\frac{U^3}{g_{_{YM}}\sqrt{Q}}(-dt^2 + dx^2) + \frac{g_{_{YM}}\sqrt{Q}}{U^3}(dU^2 + U^2 d\Omega_7^2) \right) && (2.7)\\
e^\phi &= \frac{g_{_{YM}}^3 \sqrt{Q}}{U^3} && (2.8)\\
\chi &= \frac{P}{Q} && (2.9)\\
B_{NS} &= 0 && (2.10)\\
B_{RR} &= \frac{\alpha' U^6}{g_{_{YM}}^4 Q} \ . && (2.11)
\end{aligned}$$

The metric, the dilaton, and the two-form fields are exactly the same as the near horizon limit of the $(0, Q)$ string [?]. The effect of the non-vanishing electric flux manifests itself only in the constant axion background (2.9).

The dual supergravity description of the (P, Q) gauge theory is valid in the regime of couplings and scales where both the string coupling e^ϕ and the curvature of the near-horizon geometry, measured in string units, remain small.

2.3 The NCOS decoupling limit of (P,Q) strings

Here we describe the decoupling limit of (P, Q) string theory that produces 1+1-dimensional NCOS theory, and introduce its supergravity dual. A new element in our discussion is that, as a result of including the IIB axion field, the NCOS coupling G_o^2 is incorporated as a continuous free parameter.

1+1-dimensional NCOS theory arises from the world-sheet theory on the (P, Q) string bound state in IIB theory upon taking the decoupling limit

$$g_s \to \infty, \qquad g_s^2\,\alpha' \;\text{ fixed.} \tag{2.12}$$

This limit is S-dual to the SYM limit (2.5). The NCOS parameters α'_{eff} and G_o^2 are related to the IIB parameters via

$$g_s^2\,\alpha' = G_o^4\,\alpha'_{eff}, \qquad \alpha'\mathrm{tr}F = 1 - \frac{\alpha'}{2\alpha'_{eff}}, \tag{2.13}$$

where $F = \epsilon^{01}F_{01}$ is the $U(Q)$ Yang-Mills field strength on the D-string worldsheet. In the limit (2.12), $\mathrm{tr}F$ is automatically tuned to approach its critical value $\alpha'\mathrm{tr}F = 1$, at precisely such a rate that its electrostatic force counteracts the infinite fundamental open string tension, so as to produce a finite effective tension α'_{eff} of the NCOS strings.

To see this explicitly, consider the Born-Infeld effective lagrangian of the D-string bound state (omitting all fields except the 1+1-d gauge field)

$$\mathcal{L} = -\frac{1}{g_s\alpha'}\mathrm{tr}\sqrt{(1-(\alpha' F)^2)} + \chi\mathrm{tr}F. \tag{2.14}$$

Here we included the topological term associated with the constant axion field χ. The compactness of the gauge group implies that the $U(1)$ part of the electric field $P = \mathrm{tr}E$ where E, defined as the canonical conjugate to the gauge field $E = \frac{\partial\mathcal{L}}{\partial\dot{A}}$, takes on integer values only. Inverting the relation

$$P - \chi Q = \frac{\alpha' \mathrm{tr} F}{g_s \sqrt{(1-(\alpha' F)^2)}} \tag{2.15}$$

reveals that the field strength indeed becomes near-critical in the NCOS limit (2.12)

$$\mathrm{tr} F \;\simeq\; 1 - \frac{Q^2}{2g_s^2(P-\chi Q)^2}. \tag{2.16}$$

Furthermore, from the relations (2.13) defining the NCOS parameters, we read off that

$$G_o^2 = \frac{Q}{|P-\chi Q|}. \tag{2.17}$$

As announced, the effective coupling G_o^2 can thus indeed attain arbitrary real, positive values. As we will see shortly, the existence of more general NCOS theories with a continuously varying coupling also naturally arises from S-duality symmetry of the underlying IIB string theory.

The full supergravity solution dual to the 1+1-d non-commutative open string theory, for arbitrary values of the parameters $\Big\{ g_{{}_{YM}},\, G_o,\, P,\, Q \Big\}$, follows from the general expression obtained by Schwarz in [?], by applying the NCOS scaling limit (2.12). One finds

$$\begin{aligned}
ds^2 &= \alpha' \left(1 + \frac{U^6 G_o^4}{g_{{}_{YM}}^6 Q}\right)^{1/2} \left(\frac{U^3}{g_{{}_{YM}}\sqrt{Q}}(-dx_0^2 + dx_1^2) + \frac{g_{{}_{YM}}\sqrt{Q}}{U^3}(dU^2 + U^2 d\Omega_7^2)\right), && (2.18)\\
e^{\phi} &= \frac{g_{{}_{YM}}^3 \sqrt{Q}}{U^3}\left(1 + \frac{U^6 G_o^4}{g_{{}_{YM}}^6 Q}\right), && (2.19)\\
\chi &= \frac{g_{{}_{YM}}^2 PQ + G_o^2(G_o^2 P + Q)U^6}{Q(g_{{}_{YM}}^6 Q + G_o^4 U^6)}, && (2.20)\\
B_{NS} &= -\frac{\alpha' G_o^2 U^6}{g_{{}_{YM}}^4 Q}, && (2.21)\\
B_{RR} &= \frac{\alpha'(Q + G_o^2 P)U^6}{g_{{}_{YM}}^4 PQ}\;. && (2.22)
\end{aligned}$$

Here, relative to the notation used in [?], we made the identifications

$$g_s^2 = \frac{G_o^6}{\alpha' g_{{}_{YM}}^2}, \qquad \chi_\infty = \frac{P}{Q} + \frac{1}{G_o^2}, \qquad r^2 = \frac{\alpha' G_o^2 U^2}{g_{{}_{YM}}^2}. \tag{2.23}$$

This dual supergravity description of (P,Q) NCOS theory is valid in the regime of couplings and scales where both e^{ϕ} and the curvature of the near-horizon geometry remain small.

2.4 SL(2,Z) duality of NCOS theory

In this subsection, we describe how $SL(2,\mathbf{Z})$ S-duality transformations act on the NCOS data g^2_{YM}, G^2_o, P, and Q, and write the supergravity data in a more manifestly S-duality covariant form.

Before taking any decoupling limit, the (P,Q) strings are permuted by the $SL(2,\mathbf{Z})$ S-duality symmetry of the IIB theory, via

$$\begin{pmatrix} \tilde{P} \\ \tilde{Q} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}, \tag{2.24}$$

which leaves $N = \gcd(P,Q)$ invariant. The string coupling g_s and axion χ transform via

$$\tilde{\lambda} = \frac{a\lambda + b}{c\lambda + d}, \qquad \lambda = \chi + \frac{i}{g_s}. \tag{2.25}$$

Evidently, S-duality does not preserve the super Yang-Mills decoupling limit; instead it is mapped onto the NCOS limit.

The NCOS limit, on the other hand, is in general preserved. The axion field χ_∞ transforms like

$$\tilde{\chi}_\infty = \frac{a\chi_\infty + b}{c\chi_\infty + d} \tag{2.26}$$

in the scaling limit. For generic values of χ_∞, the $SL(2,\mathbf{Z})$ transformed value is again finite, and thus the transformed theory is also a regular NCOS theory. Using the relations (2.13) and (2.17) with the IIB parameters, a straightforward calculation shows that the $SL(2,\mathbf{Z})$ transformation law for the NCOS parameters reads

$$\begin{pmatrix} \tilde{P} \\ \tilde{Q} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} \tag{2.27}$$

$$\tilde{g}^2_{YM} = \frac{g^2_{YM}(cP + dQ)^3}{Q^3}, \tag{2.28}$$

$$\tilde{G}^2_o = \frac{(cP + dQ)(cQ + (cP + dQ)G^2_o)}{Q^2}. \tag{2.29}$$

These formulas closely resemble the Morita duality transformations of the parameters of non-commutative Yang-Mills theory [**?**, **?**].

Note that, in the special case that χ and G^2_o are rational, there is always one particular $SL(2,Z)$ transformation for which the denominator in (2.26) vanishes, which implies that the transformed theory has $\tilde{G}^2_o = 0$. Hence in this case the NCOS theory can be mapped back onto a commutative SYM theory. Conversely, this means that any NCOS theory with rational G_o^2 has a precise field theoretic definition via this equivalent SYM gauge theory. NCOS theories with irrational G_o^2, on the other hand, do not have such a UV definition; they need to be defined via the corresponding IIB decoupling limit (2.12).

In order to make the $SL(2,\mathbf{Z})$ multiplet structure of the supergravity background more manifest, it is convenient to go to the Einstein frame, where the metric becomes

$$ds^2 = l_p^2 \left(\frac{U^3}{g_{_{YM}}^3\sqrt{Q}}\right)^{1/2} \left(\frac{U^3}{g_{_{YM}}\sqrt{Q}}(-f(U)dx_0^2 + dx_1^2) + \frac{g_{_{YM}}\sqrt{Q}}{U^3}(f^{-1}(U)dU^2 + U^2 d\Omega_7^2)\right). \tag{2.30}$$

Here we have included the thermal factor

$$f(U) = 1 - \frac{U_0^6}{U^6} \ . \tag{2.31}$$

This solution describes a non-extremal black string with a horizon located at at $U = U_0$. It should therefore be interpreted as the supergravity dual of generic NCOS theory at a finite temperature T_0. The relation between U_0 and T_0 can be determined by analytically continuing the solution to Euclidean signature $t = i\tau$ and looking at the metric in the U and τ coordinates near $U = U_0$. Introducing the coordinate

$$\rho^2 \sim 1 - \frac{U_0^6}{U^6} \tag{2.32}$$

the metric near $\rho = 0$ take the form

$$ds^2 \sim (d\rho^2 + \frac{9U_0^4}{g_{_{YM}}^2 Q}\rho^2 d\tau^2) \tag{2.33}$$

from which we infer that (up to factors of order one), the temperature T_0 equals

$$T_0 \simeq \frac{U_0^2}{g_{_{YM}}\sqrt{Q}} \ . \tag{2.34}$$

This is the standard UV/IR relation for D1-branes [**?**].

Since temperature is a physical notion independent of the S-duality orbit, let us now choose to parameterize the radial coordinate by

$$T = \frac{U^2}{g_{_{YM}}\sqrt{Q}} \ , \tag{2.35}$$

and introduce the $SL(2,\mathbf{Z})$ invariant combination

$$\gamma^2 = \frac{g_{_{YM}}^2}{Q^3}. \tag{2.36}$$

Replacing g^2_{YM} and U by γ^2 and T, the supergravity solution in Einstein frame becomes

$$ds^2 = l_p^2 \left(\frac{T}{\gamma}\right)^{1/4} \left(T^2(-dt^2 + dx^2) + \frac{1}{4T^2} dT^2 + d\Omega_7^2\right) \tag{2.37}$$

$$\begin{pmatrix} B_{NS} \\ B_{RR} \end{pmatrix} = l_p^2 \begin{pmatrix} -G_o^2 \\ 1 + \frac{G_o^2 P}{Q} \end{pmatrix} \left(\frac{T^3}{\gamma Q}\right) \tag{2.38}$$

$$e^\phi = \frac{\gamma^3 Q^4 + G_o^4 T^3}{\gamma^{3/2} Q^2 T^{3/2}} \tag{2.39}$$

$$\chi = \frac{\gamma^3 P Q^4 + G_o^2 (G_o^2 P + Q) T^3}{Q(\gamma^3 Q^4 + G_o^4 T^3)} . \tag{2.40}$$

As expected, we see that the Einstein frame metric is S-duality invariant. The two-form fluxes and the axion and dilaton fields, on the other hand, transform covariantly.

In the following, we will identify the dilaton profile (2.39) as a function of the radial coordinate T with the actual effective string coupling, in the given (P, Q) frame, as a function of the physical temperature. The justification for this identification is that, at temperature T_0, most of the degrees of freedom of the supergravity can be thought of as being localized near the black string horizon at $T = T_0$. Indeed, provided we are in a regime where the supergravity description is valid, we can identify the thermodynamic entropy density of the NCOS theory with the Bekenstein-Hawking entropy of the black string

$$s = \frac{S}{V} = \frac{A_H}{l_p^8 V} = \frac{T^2}{\gamma}. \tag{2.41}$$

This confirms that the NCOS matter can be thought of as forming the thermal atmosphere of the black string, and that the strength of interactions is governed by the effective string coupling e^ϕ close to the horizon at $T = T_0$.

3 Phases of NCOS theory

In the previous section, we formulated NCOS both as a decoupled theory on a brane and as a supergravity dual. As we emphasized in the introduction, these two formulations of the same theory should complement one another, in the sense that depending on the circumstances, one or the other should single itself out as the preferred description of the system. Let us address this issue concretely by first fixing all of the parameters g^2_{YM}, G_o^2, P and Q, while varying the temperature.

For starters, the value of the dilaton needs to stay small in order for the theory to be weakly coupled. From the form of the dilaton given in (2.39), we find that this restricts the temperature to take value in the range $e^\phi \ll 1$, or

$$Q^2 \left(\frac{1-\sqrt{1-4G_o^4}}{2G_o^4} \right) \ll \left(\frac{T}{\gamma} \right)^{3/2} \ll Q^2 \left(\frac{1+\sqrt{1-4G_o^4}}{2G_o^4} \right) . \tag{3.1}$$

It is clear that G_o^2 must be less than 1/2 for this region to exist, and that the region is bounded below at $T^{3/2} = \gamma^{3/2} Q^2$. We will analyze what happens outside this range of temperatures in the following section.

Let us now explore the full range of validity of the (P, Q) supergravity description. The general criteria for the effectiveness of supergravity description [**?**] dictate that the curvature radius as measured in the string frame metric should be large compared to the string length. The curvature radius of the background (2.37) can be estimated from the radius of the 7-sphere forming the black string horizon. Comparing this radius with the string length, we thus deduce that the supergravity approximation breaks down in the region

$$\frac{R^2}{\alpha'} = \left(\frac{T}{\gamma} \right)^{1/4} e^{-\phi/2} \ll 1 . \tag{3.2}$$

It can be seen that this region is completely contained inside of the range (3.1). Both regions are indicated in figure 1, for values of G_o^2 ranging from $-1/2$ to $1/2$. For the vertical coordinate in the figure we have chosen $T^{-3/2}$, so that the $e^\phi = 1$ boundary (3.1) takes the form of a circle, indicated by the black dashed line. The regime (3.2) is bounded by the red solid line, so that the supergravity description is valid in the region inside the black dotted line and outside the red solid line. Note that the ultraviolet is at the lower end of the figure.

Since the curvature radius of the supergravity is small inside the red circle, we can expect that the dual gauge theory description may take over in this region. This is most easily verified at the special vertical line at $G_o^2 = 0$, where the non-commutativity parameter is turned off. This line corresponds to ordinary 1+1 dimensional super Yang-Mills theory. The red line intersects the $G_0^2 = 0$ axis at $T^2 = g_{YM}^2/Q$, which is indeed exactly the point where the 't Hooft coupling of the gauge theory is of order one. Moreover, since SYM theory in 1+1 dimensions is super-renormalizable, the gauge theory description remains valid for arbitrarily high energies; this is indicated in the figure 1 by the fact that the red circle is touching the abscissa.

Away from $G_o^2 = 0$, the effect of non-commutativity should manifest itself. Specifically, when starting from the infra-red (from above in the figure), we expect that when the tem-

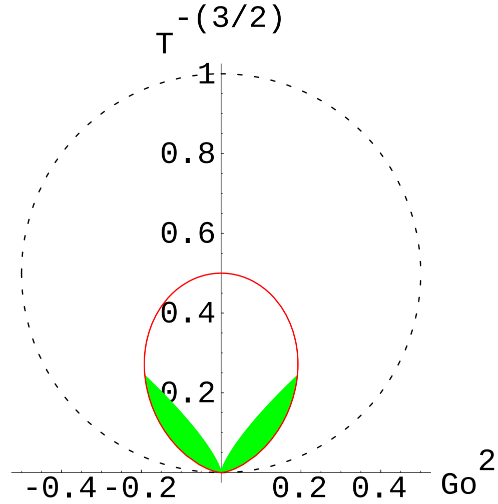


Figure 1: This figure indicates the regimes of validity of the three possible phases of NCOS theory, for given charges P and Q: (i) the supergravity phase, inside the black dashed circle, outside the red line, (ii) the gauge theory phase, inside the red line, and (iii) the matrix string phase, the green shaded region.

perature T reaches the scale set by the NCOS string tension

$$T = \frac{1}{\sqrt{\alpha'_{eff}}} = \sqrt{\frac{g_{YM}^2}{G_o^2}} \tag{3.3}$$

the theory must undergo a Hagedorn transition. Beyond this temperature, the system is most accurately described by a matrix string theory (MST) phase [**?**], that is a sigma model on $\mathbf{R}^8/S_N$ describing the eigenvalue dynamics of the matrix scalar fields of the SYM model. These matrix strings can be thought of as ionized NCOS strings, that due to the thermal fluctuations have managed to escape the D-string bound state via the Coulomb branch. A concrete quantitative check of this physical picture is provided by the fact that the effective tension of a long fundamental string that escapes to infinity in the supergravity geometry (2.18) coincides with the tension of the NCOS strings:

$$\frac{1}{2\alpha'_{eff}} = \frac{1}{\alpha'}(\sqrt{g^s_{00}g^s_{11}} + B_{NS}) = \frac{g_{YM}^2}{2G_o^2}. \tag{3.4}$$

The range of temperatures in which this phase dominates is illustrated by the green shaded region in figure 1.

At even higher temperatures, one hits the boundary of the red region where the supergravity description again becomes valid. At that temperature, the long NCOS strings recombine with the D-string due to the strong gravitational attraction caused by the black hole geometry of the supergravity dual [?]. The sequence of phases

$$\text{SUGRA} \quad \to \quad \text{NCOS} \quad \to \quad \text{MST} \quad \to \quad \text{SUGRA} \tag{3.5}$$

going up in temperature was also described in [?].

4 SL(2,Z) duality cascades

Our remaining task now is to describe what happens outside the black circle in figure 1. Clearly, since the effective string coupling is getting large there, we can expect that the system goes over into another S-dual regime. A small subtlety is that, because of the non-trivial axion background, a simple inversion will not necessarily map strong coupling to weak coupling. More general $SL(2,\mathbf{Z})$ transformations may be needed.

To address this issue in a systematic way, we will take advantage of the survey of $SL(2,\mathbf{Z})$ duality transformations of NCOS theory given in section 2. To begin, it will turn out to be convenient to exploit the S-duality equivalence and combine all theories into one single parameterization. The most convenient choice is to take as the base theory, the system with charges

$$P = 0, \qquad Q = N \tag{4.1}$$

and couplings

$$g_{_{YM}}^2, \qquad G_o^2, \tag{4.2}$$

and parameterize all the dual theories by the element

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z}) \tag{4.3}$$

which maps it to the system with couplings and charges

$$\tilde{g}_{_{YM}}^2 = d^3 g_{_{YM}}^2, \qquad \tilde{G}_o^2 = d(c + dG_o^2), \qquad \tilde{P} = bN, \qquad \tilde{Q} = dN \ . \tag{4.4}$$

In other words, we can use $g_{_{YM}}^2$ to set the scale, and G_o^2 as the data parameterizing the $SL(2,\mathbf{Z})$ equivalence class, and c and d as the data parameterizing the specific elements of the $SL(2,\mathbf{Z})$ orbit.

In terms of these data, the dilaton profile (2.39) as a function of temperature takes the following form

$$e^{\phi} = (c + dG_o^2)^2 \left(\frac{T^{3/2}}{\gamma^{3/2} N^2} \right) + d^2 \left(\frac{\gamma^{3/2} N^2}{T^{3/2}} \right) \ . \tag{4.5}$$

At this point, it is convenient to introduce the dimensionless parameters

$$x = G_o^2, \qquad y = \frac{\gamma^{3/2} N^2}{T^{3/2}} \tag{4.6}$$

to quantify the coupling and the temperature, respectively. Note, as we did in section 4, that y scales like $T^{-3/2}$ so small y corresponds to large temperature. In terms of x and y, the dilaton profile (4.6) becomes

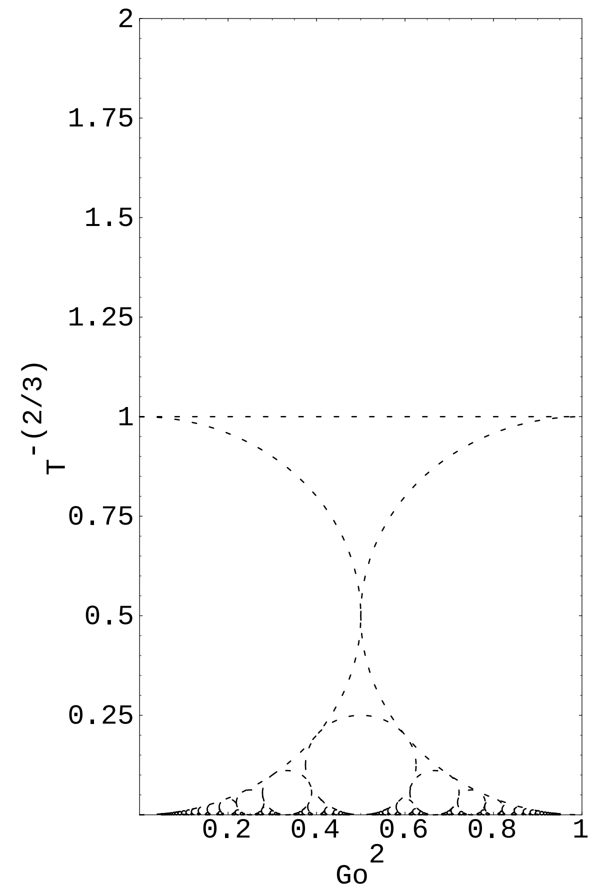
$$e^{\phi} = \frac{(c+dx)^2}{y} + d^2 y \ . \tag{4.7}$$

Our task is to determine, for given value of the parameters x and y, which effective theory provides the best description of the system. As a first step in this procedure, we will identify the pair of integers (c, d) which minimizes the string coupling (4.7) at each given point in the (x, y)-plane. For this purpose, it is helpful to first draw the locus on the (x, y)-plane for which $e^{\phi} = 1$ for all possible integers (c, d). These loci are circles and are illustrated in figure 2.a.

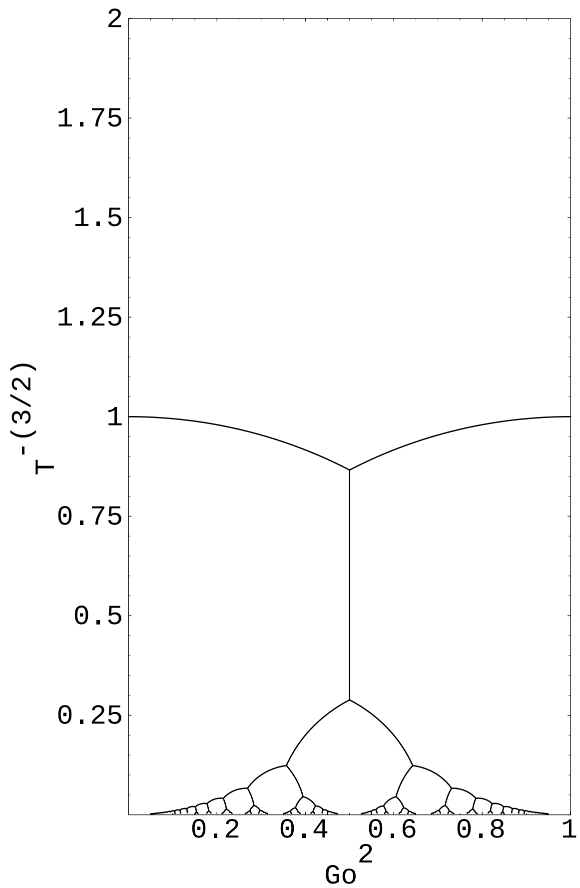
The circle corresponding to $(c, d) = (0, 1)$ is the one drawn earlier in figure 1; the rest are its generalizations to other values of (c, d). Inside each of the circles, the corresponding string coupling $g_s = e^{\phi}$ is smaller than 1. None of the circles overlap, so if a point (x, y) happens to be inside a (c, d) circle, the most weakly coupled theory is the one labeled by (c, d). There are some points, however, which are not covered by the circles. Here we can not apply $SL(2, \mathbf{Z})$ duality to make the string coupling less than one. However, since we are only interested in identifying the dual theory which minimizes the dilaton, we can take the freedom to extend the circular regions in such a way that adjoining (c, d) cells have the same value of the dilaton along the boundary. The resulting (c, d) cells, which now fill the entire (x, y) plane, are illustrated in figure 2.b.

These (c, d) regions on the phase diagram have an identical structure to that found in [**?**] in the context of non-commutative gauge theory on a torus, where the role of $SL(2, \mathbf{Z})$ was played by the Morita equivalence relation. As was emphasized in [**?**], these phase structures also bear very interesting resemblance to the phase structure of lattice spin models with θ parameters considered in [**?**, **?**]. Similar strucutres have also appeared in the context of dissipative Hosfstadter model [**?**] and quantum Hall systems [**?**].

The analysis of the phase structure in each of the (c, d) cells will closely parallel our earlier discussion in section 3. In particular, we expect that within each of the cells, we can identify three different regions, corresponding to the effective gauge theory phase, matrix string phase, and the supergravity phase. The respective ranges of validity of these different phases are summarized in the phase diagram displayed in figure 3.



(a)



(b)

Figure 2: In the left figure, we have indicated the circles in the (x, y) plane inside of which $e^{\phi} < 1$ for some integers (c, d). In the right figure, these regions are extended, such that the adjoining (c, d) cells have the same value for the dilaton along the boundary. Inside each cell, one unique (c, d) description minimizes the dilaton.

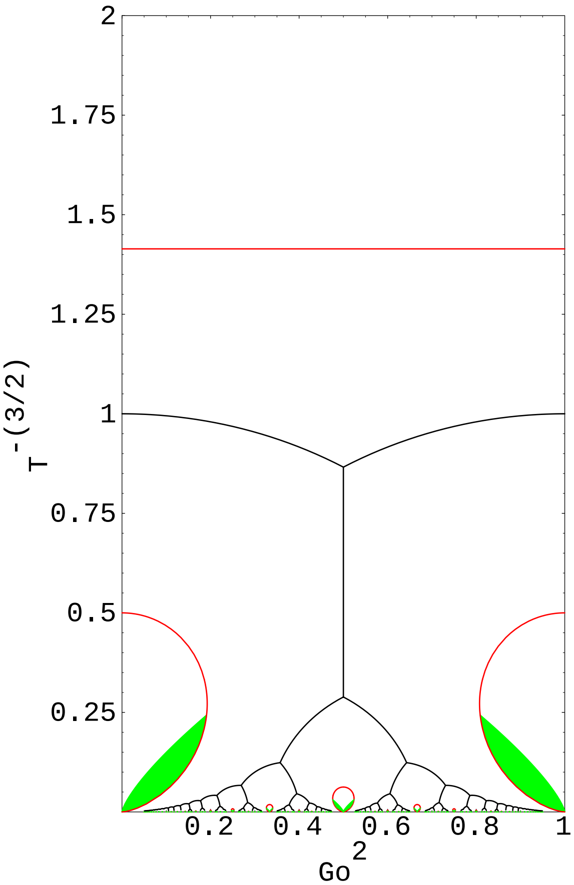


Figure 3: The phase diagram that combines all possible phases of SYM/NCOS theory in 1+1 dimensions for fixed $N = \gcd(P, Q)$. A unique (P, Q) theory provides the most weakly coupled description inside each fundamental domain. Each fundamental domain is further divided into the supergravity phase (outside the red circle), the field theory phase (inside the red circle), and the coexisting matrix string phase (shaded green region).

Let us highlight some of the features of this phase diagram.

- The vertical axis is proportional to $T^{-3/2}$, so that the ultraviolet corresponds to the bottom, and the infrared to the upper end of the figure. Each vertical slice corresponds to the $(0, N)$ NCOS theory with given G_o^2.
- For every rational value of G_o^2, there is a (c, d) cell touching the horizontal axis at $y = 0$. At this point

$$\tilde{G}_o^2 = d(c + dG_o^2) = 0, \tag{4.8}$$

  it corresponds to an ordinary super Yang-Mills theory, with gauge group $U(dN)$ and electric flux $P = bN$.
- Starting from a given SYM gauge theory with E-flux, one can flow upwards in y toward the infrared. It is possible that the system then crosses over into another (c, d) cell, and reaches another effective gauge theory phase. This effective gauge theory is deformed with an irrelevant non-commutative perturbation, proportional to the effective tension of the corresponding NCOS phase. This sequence of phases has been described in [**?**]. What we see here, however, is that the sequence of phases does not necessarily stop here: by going further toward the infrared, one can potentially cross many more (c, d) cells before reaching the deep infrared region at $y = \infty$.
- The number of $SL(2, \mathbf{Z})$ transformations involved in the flow from the UV to the IR depends sensitively on the rationality of G_o^2. Irrational values of G_o^2 require infinitely many $SL(2, \mathbf{Z})$ transformation in the ultraviolet region, as indicated by the fractal phase pattern illustrated in figure 3.
- Since the ionization/recombination phase transition associated with the Hagedorn scale takes place in each of the (c, d) cells, the system can undergo these transitions multiple times as the temperature is varied monotonically.

As a concrete illustration of this type of duality cascade, let us consider a given theory with parameters

$$P = 0, \qquad Q = N, \qquad G_o^2 = \frac{1}{n_1 - \frac{1}{n_2}} \tag{4.9}$$

corresponding in the ultra-violet to a $U(N(n_1 n_2 - 1))$ super Yang-Mills theory with Nn_1 units of electric flux. The SYM degrees of freedom, however, are weakly coupled in the far UV only. In particular, since electric flux creates a mass gap in two dimensions, one expects that towards the infra-red, the $U(Q)$ gauge symmetry gets broken to $U(N)$. Ultimately, the system will flow towards $U(N)$ matrix string theory.

It is instructive to trace all the intermediate phases between the UV gauge theory phase and the IR matrix string phase. They are listed in figure 4, where we have also indicated the behavior of the entropy in all different phases. It will turn out that the phases are well separated provided that $n_2 \gg Nn_1 \gg N^2 \gg 1$.

Starting from the infra-red, flowing down towards the UV, the system first follows the successive phases outlined in [?] and [?]. Continuing further towards the UV, however, the theory again enters a supergravity phase. At the point where $e^\phi = 1$, we now need to apply the S-duality transformation

$$\begin{pmatrix} 0 & 1 \\ -1 & n_1 \end{pmatrix}, \tag{4.10}$$

connecting to an NCOS theory with charges (N, Nn_1). Then, after a similar sequence of supergravity, gauge theory, and matrix string theory phases, the duality cascade continues via the S-duality transformation

$$\begin{pmatrix} 0 & 1 \\ -1 & n_2 \end{pmatrix} \tag{4.11}$$

which finally takes us to a commutative theory with charges $(P, Q) = (Nn_1, N(n_1 n_2 - 1))$.

5 Conclusions

In this article, we investigated the full phase structure of non-commutative open string theories in 1+1 dimensions and found a rich fractal structure closely resembling the phase structure of non-commutative gauge theory on a torus. The most striking conclusion of our analysis is that, with increasing temperature, the system can undergo multiple transitions between alternating gauge theory, matrix string theory, and supergravity phases.

A comment is in order regarding the nature of the thermodynamic transitions at the various phase boundaries. Since we are working in 1+1 dimensions, strictly speaking there should not be any phase transition, unless possibly when we take the large N limit. At large but finite N, it is more appropriate to refer to the transitions between the different phases as "crossovers." It is conceivable, however, that in the $N \to \infty$ limit, some of the crossovers (in particular the Hagedorn transition) may actually become true phase transitions. Even without sharp transitions, however, the duality cascade described in this article should have many observable consequences.

For rational values of G_o^2, the thermodynamics described here is that of ordinary super Yang-Mills theory with some electric flux. It would be interesting to see if it is possible to

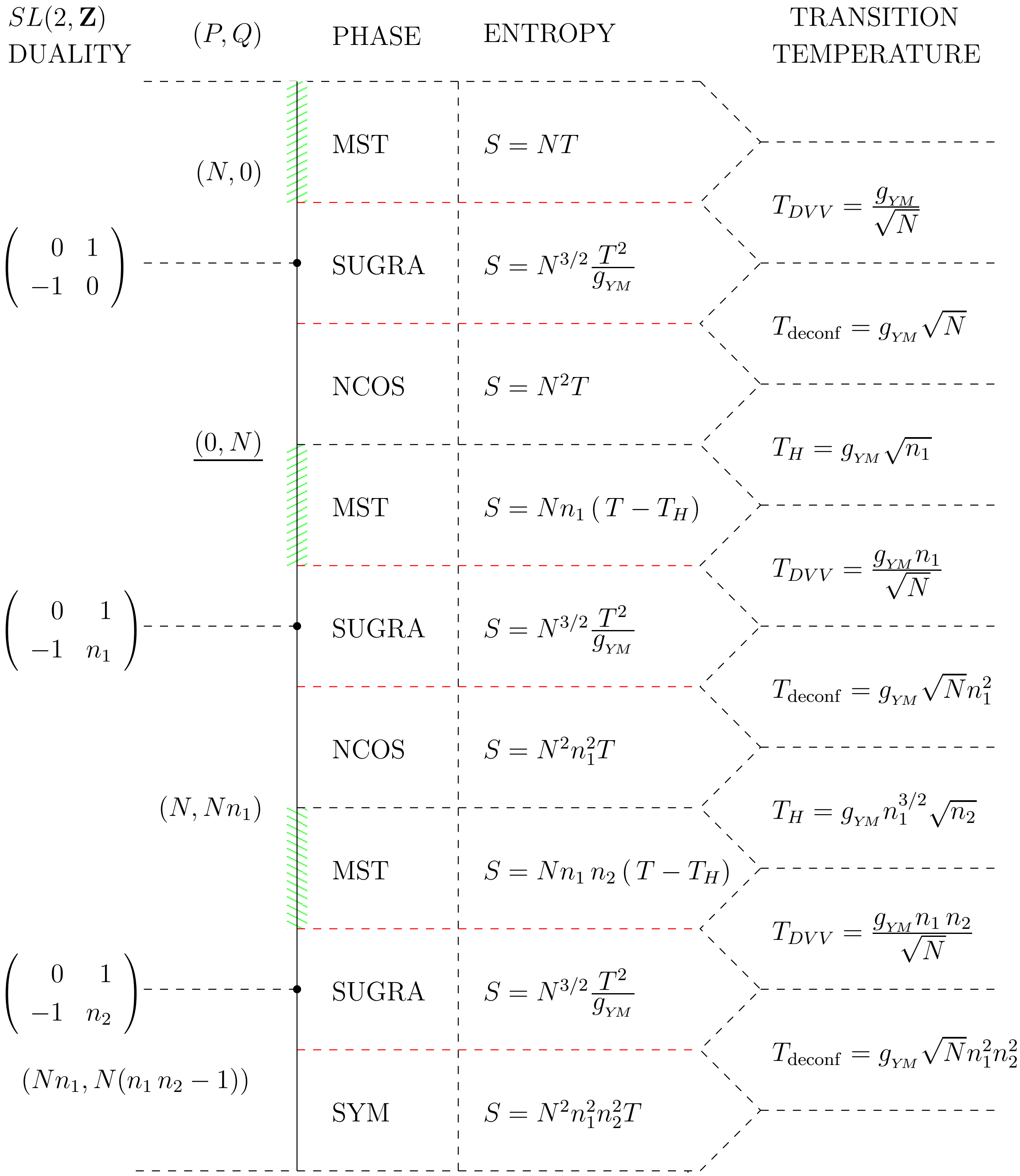


Figure 4: An overview of the duality cascade and all the intermediate phases for the special case of a $(0,N)$ NCOS theory with $G_o^2 = \frac{1}{n_1 - \frac{1}{n_2}}$ with $n_2 \gg Nn_1 \gg N^2 \gg 1$. We have also given the qualitative behavior of the entropy, and the transition temperatures. The entropy is maximal in the ultraviolet (bottom of the figure), and decreases monotonically towards the infrared (top of the figure).

reproduce some of our results more directly via other techniques. For example, it may be possible to find signatures of these phase transitions in the behavior of the thermal partition function of the gauge theory or of the two-point function of the stress energy tensor [**?**]. Perhaps a computation on the lattice or DLCQ methods [**?**, **?**] can provide some useful insights.

Much of the $SL(2, \mathbf{Z})$ structure of NCOS theory in 1+1 dimension can rather straightforwardly be generalized to higher dimensions. There are several important differences, however. In 3+1 dimensions, for example, g_{YM}^2 is a freely adjustable dimensionless quantity. Therefore, the full phase diagram will be three dimensional, parameterized by g_{YM}^2, G_o^2, and T (measured in units of the NCOS string length). A preliminary study indicates that the cross sections with constant g_{YM}^2 display an analogous structure as describe here, whereas the cross sections for constant G_0^2 look similar as the phase diagram described in [**?**] for the case of small G_o^2. It should be instructive to map out the full multi-dimensional phase structure also for these higher dimensional cases.

Another interesting open question about the 1+1-dimensional case is what happens in the case of irrational G_o^2. In this case the system does not have any known UV definition. Nonetheless, at any finite temperature, it can be approximated to arbitrary precision by a sequence of rational G_o^2 theories, which do have a UV description. It would be very interesting to find out whether the irrational theory allows for an independent UV fixed point description.

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